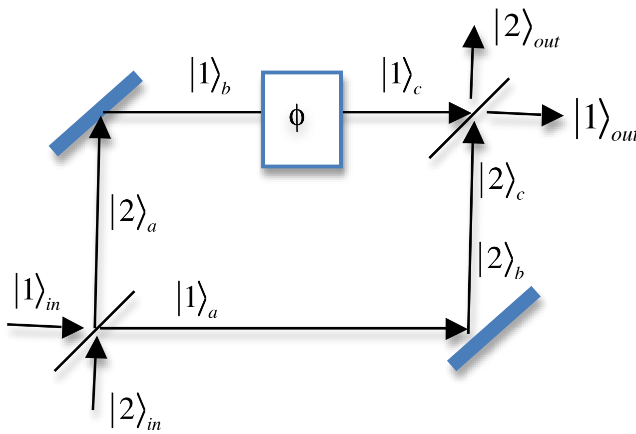
Physics 566 Fall 2010
Problem Set #1
Due: Friday Sep. 3, 2010

Problem 1: Equivalence of Mach-Zender and SU(2) interferometer

A Mach-Zender interferometer in optics is sketched below.



The 50-50 beam splitters are thin black lines and the mirrors are thick blue lines. We assume here that the optical path lengths of the two arms of the interferometer are equal up to a phase-shift ϕ in the upper arm. A single photon sent to the interferometer can be in one of two modes. The quantum system is thus described in a two-dimensional Hilbert space, isomorphic to a spin-1/2 particle.

(a) Show that up to a negligible overall phase, the sequence of SU(2) transformations

$$\pi/2\text{-}x\text{-rotation} \rightarrow \pi\text{-}x\text{-rotation} \rightarrow \phi\text{-}z\text{-rotation} \rightarrow \pi/2\text{-}x\text{-rotation}$$

yields what you would expect for the transformation on the two modes

beam-splitter → mirrors → phase shift →beam-splitter

(b) Sketch the sequence of transformations on the Bloch sphere starting at the south pole.

(c) Assuming the photon enters in $|1\rangle_{in}$, show that the probability to find the photon in $|1\rangle_{out}$ after passing through the interferometer is $P_{1_{out}} = \cos^2\left(\frac{\phi}{2}\right)$.

(d) Repeat part (a) for the following mixed state inputs

$$(i)\ \hat{\rho}_{in} = \frac{1}{2}|1_{in}\rangle\langle 1_{in}| + \frac{1}{2}|2_{in}\rangle\langle 2_{in}|;\ \ (ii)\ \hat{\rho}_{in} = \frac{1}{3}|1_{in}\rangle\langle 1_{in}| + \frac{2}{3}|2_{in}\rangle\langle 2_{in}|.$$

Comment on your results.

Problem 2: Different ensemble decompositions - example spin 1/2/

(a) Suppose we have a statistical mixture of spin 1/2 particles that consists of the state $|+_z\rangle$ mixture with probability $\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)$ and the state $|-_z\rangle$ with probability $\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)$.

Find the matrix of the density operator in the basis $\{|+_z\rangle, |-_z\rangle\}$, and in the basis of eigenstates of $\hat{\sigma}_x$, $\{|+_x\rangle, |-_x\rangle\}$. What is the Bloch vector that describes this state?

(b) Now suppose we have an mixed state with 1/2 probability to have spin along $\mathbf{e}_{n_1} = \frac{1}{\sqrt{2}}(\mathbf{e}_z + \mathbf{e}_x)$ and 1/2 probability to have spin along $\mathbf{e}_{n_2} = \frac{1}{\sqrt{2}}(\mathbf{e}_z - \mathbf{e}_x)$. Is this a completely mixed state? Write the density operator in the basis $\{|+_z\rangle, |-_z\rangle\}$. Compare to part (b). Please comment on your result.

(c) Show that two statistical mixtures of pure states, $\{|+_n\rangle\}$ with probabilities p_n, and $\{|+_m\rangle\}$ with probabilities q_m, describe the *same* density operator $\hat{\rho}$ if

$$\mathbf{Q} = \sum_n p_n \mathbf{e}_n = \sum_m q_m \mathbf{e}_m ,$$

where $\mathbf{Q}$ is the Bloch vector of $\hat{\rho}$. Check this with your results of parts (b) and (c).

Problem 3: Ambiguity of ensemble decompositions of density operators

We saw in Problem 2 that a density operator does not decompose uniquely into a statistical mixture of pure states. This has profound implications for both practical calculations of the density matrix (as we will see later in the semester) as well as for foundational descriptions of states in quantum mechanics.

What different ensemble are possible to yield a given density operator? In this problem we prove the following.

<u>Hamilton-Jozsa-Wootters theorem:</u> The two density operators

$$\hat{\rho}_1 = \sum_i p_i |\psi_i\rangle\langle\psi_i| \text{ and } \hat{\rho}_2 = \sum_j q_j |\phi_j\rangle\langle\phi_j|$$

are equal if and only if the two ensembles are related by,

$$\sqrt{q_j}|\phi_j\rangle = \sum_i U_{ji}\sqrt{p_i}|\psi_i\rangle,$$

where U_{ji} are elements of a unitary matrix.

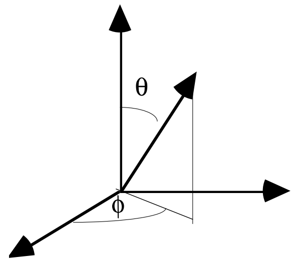
(a) Assume the relation between the ensembles is true. Prove that $\hat{\rho}_1 = \hat{\rho}_2$.

(b) Assume $\hat{\rho}_1 = \hat{\rho}_2 \equiv \hat{\rho}$. Show $\sqrt{q_j}|\phi_j\rangle = \sum_i U_{ji}\sqrt{p_i}|\psi_i\rangle$.

(Hint: Show first that $\sqrt{p_i}|\psi_i\rangle = \sum_\alpha M_{j\alpha}\sqrt{\lambda_\alpha}|e_\alpha\rangle$, where λ_α are the eigenvalues of $\hat{\rho}$ and $|e_\alpha\rangle$ its orthonormal eigenvectors and $M_{j\alpha}$ are elements of a unitary matrix. The same thus holds for $\sqrt{q_j}|\phi_j\rangle$. The proof will follow).

Problem 4: Some properties of spin 1/2 and the Bloch sphere.

Given a unit vector $\vec{\mathbf{e}}_n$, defined by angles θ and ϕ with respect to the polar axis z,



(a) Show that every pure state for a two-level system, $|\psi\rangle = \alpha|+_z\rangle + \beta|-_z\rangle$, is equivalent to a ket $|+_n\rangle$, defined as the spin-up state along an axis $\vec{\mathbf{e}}_n$. What are the angles θ and ϕ?

(b) Show that the one dimensional projector corresponding to measurement of $|+_n\rangle$ is,

$$|+_n\rangle\langle +_n| = \frac{1}{2}\left(\hat{1} + \vec{\mathbf{e}}_n \cdot \hat{\vec{\sigma}}\right).$$

(c) Show that the inner product between any two pure states is, $\left|\langle +_n | +_{n'}\rangle\right| = \cos(\Theta/2)$, where Θ is the angle between the directions $\vec{\mathbf{e}}_n$ and $\vec{\mathbf{e}}_{n'}$ in three dimensional space.

(d) Consider the polarization of a photon as a two state system. If we make the association,

$|+_z\rangle \Rightarrow$ right hand circular (positive helicity)
$|-_z\rangle \Rightarrow$ left hand circular (negative helicity)

to what polarization states do you associate $|\pm_x\rangle$ and $|\pm_y\rangle$?

The Bloch sphere description of the polarization is know as the "Poincaré sphere", with each point on the surface representing a possible elliptical polarization. The three Cartesian coordinates of the Bloch vector are also known as the "Stokes parameters". In the case of the photon, the Bloch vector actually represents the direction of its spin angular momentum.